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HOTINE'S (ω, φ, N) COORDINATE SYSTEM

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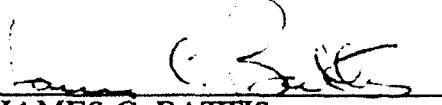
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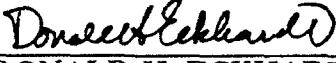
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This report is an extensive reworking of the material in Chapter 12 of Hotine's treatise Mathematical Geodesy (U.S. Department of Commerce, Washington, D.C. 1969) which dealt with the basic properties of the (ω, φ, N) coordinate system. In 1990, the author showed that Hotine's derivation of this coordinate system was incomplete, and the present document contains the full details of that work as well as the material required to rigorously complete Hotine's analysis. The latter material has not been previously published.

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CONTENTS

Preface

The following report is a working paper on Hotine's (ω, ϕ, N) coordinate system in which the details missing from Hotine's analysis have been supplied. The manuscript was started in 1989/90 and essentially represents the effort involved in my reworking of Chapter 12 of Hotine's treatise. Part of it appeared in my paper "The assertion of Hotine on the integrability conditions in his general (ω, ϕ, N) coordinate system" (*Manuscripta Geodaetica* 15 , 1990), however the major part of the document has not been previously published. Over the years I have found the material in this original, informal write-up to be a helpful reference for my ongoing research in differential geodesy. It is hoped that this slightly edited version will be useful to other theoretical geodesists.

"HOTINE'S (ω, ϕ, N) COORDINATE SYSTEM"

Joseph Zund

Summary:

This report is an extensive reworking of the material in Chapter 12 of Hotine's treatise Mathematical Geodesy (U.S. Department of Commerce, Washington, D.C. 1969) which dealt with the basic properties of the (ω, ϕ, N) coordinate system. In 1990, the author showed that Hotine's derivation of this coordinate system was incomplete, and the present document contains the full details of that work as well as the material required to rigorously complete Hotine's analysis. The latter material has not been previously published.

1 Introduction

In this paper we consider the properties of the (ω, ϕ, N) coordinate system which was the major coordinate system investigated by Hotine in his treatise [1]. It is, apart from a minor change in notation, the local astronomical coordinate system introduced by Marussi in [2], [3]. The relevance of this coordinate system for differential geodesy is often misunderstood and overestimated. In fact, the basic significance of this system is that we have it, and that it was extensively employed by Marussi and Hotine in their work. It is the obvious choice of a coordinate system which uses the Marussi Ansatz of identifying the first two coordinates with surface coordinates and the third one with the geopotential function. Moreover, to a great extent this system satisfies the conditions laid down by Marussi, [4], [5], for intrinsic geodesy. On the other hand, physically speaking, it is rigidly tied to astronomical measurements, which while of immediate importance in the time of Marussi and Hotine, no longer has a predominant role in modern geodetic investigations. One would be inclined to regard this system as merely being a historical relic, except for the cogent and embarrassing fact that we know of no other equally simple and workable coordinate system in which to check our ideas and cite as an illustrative example. A related question is to determine whether the properties of the (ω, ϕ, N) system are in any sense indicative of properties which can reasonably be expected to hold in more general coordinate systems.

The purpose of the present paper is to explicitly address and resolve these questions. In doing so, we present a new and simplified derivation of this system which allows the basic steps of its construction to be immediately seen and assessed as to their generality and speciality. Section 2 contains a review of the required preliminaries from the leg calculus including the *basic equations*, while Section 3 is concerned with the *geometrical equations* which form the first step in the construction of the (ω, ϕ, N) system. Section 4 is devoted to the *primary equations* which lead to the so-called *ω-degeneracy* which was discovered in [6]. The covariant components of the leg vectors and the line element ds^2 in the (ω, ϕ, N)-system are derived in Section 5. It is then shown in Section 6 that the (ω, ϕ, N) system automatically satisfies the Lamé equations in $\mathbf{E}_3$. These equations play the rôle of *consistency equations* which must be satisfied in order to insure that this system is well-defined.

Throughout our discussion we will frequently make reference to equations appearing in Hotine's treatise, [1], and such equation and page numbers will be indicated in square brackets.

2 The Leg Calculus

In this section we review the principal notions of the leg calculus as employed in [6], [7] and in a more rudimentary form by Hotine in his treatise, [1]. A complete development of the general theory may be found in [8]. By a 3-*leg* we mean a linearly independent system of vectorial quantities in $\mathbf{E}_3$. For our present purpose we choose

our 3-leg to be orthonormal and consist of three vectors $\boldsymbol{\lambda}$, $\boldsymbol{\mu}$, $\boldsymbol{\nu}$ which may be regarded as tangent vectors having contravariant components λ^r, μ^r, ν^r respectively, or dually three Pfaffian forms (exterior differential 1-forms)

$$\theta_1 := \lambda_r dx^r, \ \theta_2 := \mu_r dx^r, \ \theta_3 := \nu_r dx^r \tag{2.1}$$

where λ_r, μ_r, ν_r are respectively the covariant components of $\boldsymbol{\lambda}$, $\boldsymbol{\mu}$, $\boldsymbol{\nu}$, and x^r is a set of arbitrary ambient coordinates in $\mathbf{E}_3$. Abstractly, both $\left\{\boldsymbol{\lambda},\boldsymbol{\mu},\boldsymbol{\nu}\right\}$ and $\{\theta_1,\theta_2,\theta_3\}$ can be considered as isomorphic representations of the same 3-leg system.

Let $\mathbf{S}$ be an arbitrary smooth surface in $\mathbf{E}_3$ and we chose $\boldsymbol{\lambda}$, $\boldsymbol{\mu}$, to be unit tangent vectors to $\mathbf{S}$ and $\boldsymbol{\nu}$ being the unit tangent vector to a congruence of curves Γ normal to $\mathbf{S}$, i.e. $\boldsymbol{\nu}$ is a unit normal to $\mathbf{S}$. We associate to the 3-leg, nine leg parameters defined by the following contractions of the covariant derivatives with pairs of tangent vectors:

$$\begin{aligned}
k_1 &:= \lambda_{rs}\nu^r\lambda^s &= -\nu_{rs}\lambda^r\lambda^s \\
k_2 &:= \mu_{rs}\nu^r\mu^s &= -\nu_{rs}\mu^r\mu^s \\
\gamma_1 &:= \nu_{rs}\lambda^r\nu^s &= -\lambda_{rs}\nu^r\nu^s \\
\gamma_2 &:= \nu_{rs}\mu^r\nu^s &= -\mu_{rs}\nu^r\nu^s \\
t_1 &:= \mu_{rs}\nu^r\lambda^s &= -\lambda_{rs}\mu^r\lambda^s \\
t_2 &:= \nu_{rs}\lambda^r\mu^s &= -\lambda_{rs}\nu^r\mu^s \\
\sigma_1 &:= \lambda_{rs}\mu^r\lambda^s &= -\mu_{rs}\lambda^r\lambda^s \\
\sigma_2 &:= \lambda_{rs}\mu^r\mu^s &= -\mu_{rs}\lambda^r\mu^s \\
\varepsilon_3 &:= \lambda_{rs}\mu^r\nu^s &= -\mu_{rs}\lambda^r\nu^s
\end{aligned} \tag{2.2}$$

By virtue of these expressions the covariant derivatives of the leg vectors admit the following canonical leg-representations:

$$\begin{aligned}
\lambda_{rs} &= \sigma_1\mu_r\lambda_s + \sigma_2\mu_r\mu_s + \varepsilon_3\mu_r\nu_s + k_1\nu_r\lambda_s - t_2\nu_r\mu_s - \gamma_1\nu_r\nu_s, \\
\mu_{rs} &= -\sigma_1\lambda_r\lambda_s - \sigma_2\lambda_r\mu_s - \varepsilon_3\lambda_r\nu_s + t_1\nu_r\lambda_s + k_2\nu_r\mu_s - \gamma_2\nu_r\nu_s, \\
\nu_{rs} &= -k_1\lambda_r\lambda_s + t_2\lambda_r\mu_s + \gamma_1\lambda_r\nu_s - t_1\mu_r\lambda_s - k_2\mu_r\mu_s + \gamma_2\mu_r\nu_s
\end{aligned} \tag{2.3}$$

which are called the *basic leg equations* since they play a prominent role in the leg calculus. The leg parameters appearing in (2.3) have the following geometric significance where the numerical subscripts refer to the $\left(\boldsymbol{\lambda},\boldsymbol{\mu}\right)$ directions: (k_1,k_2) are the *normal curvatures* of $\mathbf{S}$; (γ_1,γ_2) are the *tendencies* of Γ; hence $\chi := \sqrt{\gamma_1^2 + \gamma_2^2}$ is the *curvature* of Γ; (σ_1,σ_2) and (t_1,t_2) are respectively the *geodesic curvatures* and *torsions* of surface curves on $\mathbf{S}$, and since $t_1 + t_2 = 0$ we may eliminate t_2 from our analysis. This parameter ε_3, which was not employed by Hotine, is more complicated: it is a measure of the deviation of $\mathbf{S}$ from being automatically a member of a triply orthogonal system of surfaces and it also is part of the torsion τ of Γ. Finally, we note that the *Gauss* (mean) and *Germain* (mean) *curvatures* of $\mathbf{S}$ are given by

$$K := k_1 k_2 - t_1^2, \ H = (k_1 + k_2)/2 \tag{2.4}$$

respectively.

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Upon introducing the absolute derivative, denoted by D, equations (2.3) become:

$$
\begin{aligned}
D\lambda_r &= (\sigma_1\mu_r + k_1\nu_r)\theta_1 + (\sigma_2\mu_r + t_1\nu_r)\theta_2 + (\varepsilon_3\mu_r - \gamma_1\nu_r)\theta_3, \\
D\mu_r &= (-\sigma_1\lambda_r + t_1\nu_r)\theta_1 + (-\sigma_2\lambda_r + k_2\nu_r)\theta_2 + (-\varepsilon_3\lambda_r - \gamma_2\nu_r)\theta_3, \\
D\nu_r &= (-k_1\lambda_1 - t_1\mu_r)\theta_1 + (-t_1\lambda_r - k_2\mu_r)\theta_2 + (\gamma_1\lambda_r + \gamma_2\mu_r)\theta_3.
\end{aligned} \tag{2.5}
$$

The first set of *Cartan's structural equations* are given by

$$
\begin{aligned}
d\theta_1 + \omega_{12} \wedge \theta_2 + \omega_{13} \wedge \theta_3 &= 0, \\
d\theta_2 + \omega_{21} \wedge \theta_1 + \omega_{23} \wedge \theta_3 &= 0, \\
d\theta_3 + \omega_{31} \wedge \theta_1 + \omega_{32} \wedge \theta_2 &= 0;
\end{aligned} \tag{2.6}
$$

hence, the *connection 1-forms* $\omega_{ab} = -\omega_{ba}$ are

$$
\begin{aligned}
\omega_{12} &= -\sigma_1\theta_1 - \sigma_2\theta_2 - \varepsilon_3\theta_3, \\
\omega_{31} &= k_1\theta_1 + t_1\theta_2 - \gamma_1\theta_3, \\
\omega_{23} &= -t_1\theta_1 - k_2\theta_2 + \gamma_2\theta_3;
\end{aligned} \tag{2.7}
$$

or

$$
\begin{aligned}
d\theta_1 &= \sigma_1\theta_1 \wedge \theta_2 - k_1\theta_3 \wedge \theta_1 + (t_1 - \varepsilon_3)\theta_2 \wedge \theta_3, \\
d\theta_2 &= \sigma_2\theta_1 \wedge \theta_2 - (t_1 + \varepsilon_3)\theta_3 \wedge \theta_1 + k_2\theta_2 \wedge \theta_3, \\
d\theta_3 &= \gamma_1\theta_3 \wedge \theta_1 - \gamma_2\theta_2 \wedge \theta_3.
\end{aligned} \tag{2.8}
$$

Since $\mathbf{E}_3$ is a flat space, the second set of *Cartan's structural equations* reduces to

$$
\begin{aligned}
d\omega_{12} &= -\omega_{31} \wedge \omega_{23}, \\
d\omega_{31} &= \omega_{12} \wedge \omega_{23}, \\
d\omega_{23} &= -\omega_{12} \wedge \omega_{31},
\end{aligned} \tag{2.9}
$$

because the curvature 2-forms $\Omega_{ab} = -\Omega_{ba}$ are identically zero. These equations, and leg differential equations defined by them, are the leg-theoretic analogues of the classical Lamé equations.

A basic notion of the leg calculus is to systematically resolve all expressions, e.g. such as (2.3) into their leg representations. For derivatives of an arbitrary smooth function F (with the leg parameters given in (2.2)) this involves the so called *Pfaffian derivatives* defined by

$$
dF := F_{/1}\theta_1 + F_{/2}\theta_2 + F_{/3}\theta_3 \tag{2.10}
$$

where $F_{/1} := \lambda^r F_r$, $F_{/2} := \mu^r F_r$, $F_{/3} := \nu^r F_r$ denote *directional* (leg) *derivatives* in the respective $\boldsymbol{\lambda}$, $\boldsymbol{\mu}$, $\boldsymbol{\nu}$ directions, and for scalar functions we follow Hotine's practice of merely adding a subscript to denote (ordinary) partial differentiation. Note that, unlike partial derivatives, the leg derivatives are generally not permutable, e.g. $F_{/1/2} - F_{/2/1} \neq 0$ etc.

The integrability conditions of the Pffafian expression are given by $d(dF) := d^2 F = 0$ which upon expansion yields

$$
\begin{aligned}
dF_{/1} \wedge d\theta_1 \; + \; & F_{/1}d\theta_1 + dF_{/2} \wedge \theta_2 + F_{/2}d\theta_2 \\
+ \; & dF_{/3} \wedge \theta_3 + F_{/3}d\theta_3 = 0
\end{aligned} \tag{2.11}
$$

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where $d\theta_1$, $d\theta_2$, $d\theta_3$ are given by (2.6) and

$$
\begin{aligned}
dF_{/1} &:= F_{/1/1}\theta_1 + F_{/1/2}\theta_2 + F_{/1/3}\theta_3 \\
dF_{/1} &:= F_{/2/1}\theta_1 + F_{/2/2}\theta_2 + F_{/2/3}\theta_3 \ . \\
dF_{/3} &:= F_{/3/1}\theta_1 + F_{/3/2}\theta_2 + F_{/3/3}\theta_3
\end{aligned}
\tag{2.12}
$$

Explicit evaluation of (2.11) then yields

$$
\begin{aligned}
&\left\{ F_{/2/1} - F_{/1/2} + F_{/1}\sigma_1 + F_{/2}\sigma_2 \right\} \theta_1 \wedge \theta_2 \\
&+ \left\{ F_{/1/3} - F_{/3/1} - F_{/1}k_1 - F_{/2}\left(t_1 + \varepsilon_3\right) F_{/3}\gamma_1 \right\} \theta_3 \wedge \theta_1 \\
&+ \left\{ F_{/3/2} - F_{/2/3} + F_{/1}\left(t_1 - \varepsilon_3\right) + F_{/2}k_2 - F_{/3}\gamma_2 \right\} \theta_2 \wedge \theta_3 = 0
\end{aligned}
\tag{2.13}
$$

and by the linear independence of the exterior products $\theta_1 \wedge \theta_2$, $\theta_3 \wedge \theta_1$, and $\theta_2 \wedge \theta_3$ we have the so called *F-commutators* given in [6]:

$$
\begin{aligned}
F_{/1/2} - F_{/2/1} &= \sigma_1 F_{/1} + \sigma_2 F_{/2}, \\
F_{/3/1} - F_{/1/3} &= -k_1 F_{/1} - \left(t_1 + \varepsilon_3\right) F_{/2} + \gamma_1 F_{/3}, \\
F_{/2/3} - F_{/3/2} &= \left(t_1 - \varepsilon_3\right) F_{/1} + k_2 F_{/2} - \gamma_2 F_{/3},
\end{aligned}
\tag{2.14}
$$

which are denoted respectively by (F_I), (F_{II}) and (F_{III}). In practice these permutability rules for the leg derivatives need not be remembered since they may be *automatically* obtained by computing $d^2F = 0$. It should be observed that all the leg equations given in this section are *independent* of the choice of a particular ambient coordinate system x^r in $\mathbf{E}_3$. In Section 3, we will consider the consequences of adapting the leg calculus to the special (ω, ϕ, N) coordinate system in $\mathbf{E}_3$.

3 Geometric Equations

The first step in adapting the leg calculus to the (ω, ϕ, N)-system is to give a proper definition of the surface coordinates (ω, ϕ) which constitutes the first part of the Marussi Ansatz, viz.

$$
x^1 := \omega, \ x^2 := \phi.
$$

The second part of this Ansatz, i.e. $x^3 := N$ where N is the geopotential function will be considered in Section 6. In order to introduce (ω, ϕ), Hotine [12.002] takes a constant orthonormal Cartesian 3-leg $\{\mathbf{A}, \mathbf{B}, \mathbf{C}\}$ whose vectors are respectively aligned along the Cartesian coordinate axes $y^r := (x, y, z)$ at a fixed origin 0 in $\mathbf{E}_3$. Then he chooses the general variable 3-leg $\left\{\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\nu}\right\}$ roughly in the directions of the parallel, meridian, and zenith at a point on the surface S. Thus, as in Section 2, $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are tangents to $\mathbf{S}$ while $\boldsymbol{\nu}$ is tangent to Γ which when N is taken to be an equipotential surface, is along the plumblines of the N-surface $\mathbf{S}$. The surface coordinates (ω, ϕ) are respectively called the *longitude* ω and the *latitude* ϕ, and are defined by the equations [12.003-.005]

$$
\begin{aligned}
\cos\omega \cos\phi &:= \nu^r A^r, \\
\sin\omega \cos\phi &:= \nu^r B^r, \\
\sin\phi &:= \nu_r C^r,
\end{aligned}
\tag{3.1}
$$

4

where tentatively we take $-\pi/2 \leq \phi \leq \pi/2$ and $0 \leq \omega \leq \pi$. Then by inspection, we have the following *algebraic geometrical equations*:

$$
\begin{aligned}
\lambda_r &= -A_r \sin\omega + B_r \cos\omega \\
\mu_r &= -A_r \cos\omega \sin\phi - B_r \sin\omega \cos\phi + C_r \cos\phi, \\
\nu_r &= A_r \cos\omega \cos\phi + B_r \sin\omega \cos\phi + C_r \sin\phi;
\end{aligned}
\tag{3.2}
$$

i.e. [12.008]. Since the matrix of the coefficients on the right hand side of (3.2) is an orthogonal matrix the *inverse geometrical equations* are given by

$$
\begin{aligned}
A_r &= -\lambda_r \sin\omega - \mu_r \cos\omega \sin\phi + \nu_r \cos\omega \cos\phi, \\
B_r &= \lambda_r \cos\omega - \mu_r \sin\omega \sin\phi + \nu_r \sin\omega \cos\phi, \\
C_r &= \mu_r \cos\phi + \nu_r \sin\phi.
\end{aligned}
\tag{3.3}
$$

Covariant differentiation of (3.2), and using (3.3) then yield the *differential* version of the *geometric equations*:

$$
\begin{aligned}
\lambda_{rs} &= (\mu_r \sin\phi - \nu_r \cos\phi)\,\omega_s \\
\mu_{rs} &= -\lambda_r \omega_s \sin\phi - \nu_r \phi_s \\
\nu_{rs} &= \lambda_r \omega_s \cos\phi + \mu_r \phi_s,
\end{aligned}
\tag{3.4}
$$

i.e. [12.014-.016], where ω_s and ϕ_s denote the gradients of ω and ϕ. An alternate derivation of (3.4) may be obtained by covariant differentiation of (3.3). Since

$$
A_{rs} = 0, \; B_{rs} = 0, \; C_{rs} = 0;
\tag{3.5}
$$

which yields the following system of equations:

$$
\begin{aligned}
\mu_{rs} \cos\phi \; + \; & \nu_{rs} \sin\phi = (\mu_r \sin\phi - \nu_r \cos\phi)\,\phi_s, \\
\lambda_{rs} \sin\omega \; + \; & \mu_{rs} \cos\omega \sin\phi - \nu_{rs} \cos\omega \cos\phi \\
= \; & [-\lambda_r \cos\omega + (\mu_r \sin\phi - \nu_r \cos\phi) \sin\omega]\,\omega_s \\
- \; & [(\mu_r \cos\phi + \nu_r \sin\phi) \cos\omega]\,\phi_s, \\
\lambda_{rs} \cos\omega \; - \; & \mu_{rs} \sin\omega \sin\phi + \nu_{rs} \sin\omega \cos\phi \\
= \; & [-\lambda_r \cos\omega + (\mu_r \sin\phi - \nu_r \cos\phi) \sin\omega]\,\omega_s \\
+ \; & [(\mu_r \cos\phi + \nu_r \sin\phi) \sin\phi]\,\phi_s.
\end{aligned}
\tag{3.6}
$$

The matrix of coefficients of the covariant derivatives on the left hand side of (3.6) is again an orthogonal matrix, so one can immediately invert this system of equations to yield (3.4).

Equations (3.4) appear — at first glance — to be quite reasonable, however, they contain a remarkable feature as will be discussed in Section 4.

4 Primary Equations

Contraction of the differential version of the geometrical equations gives the following expressions for the leg derivatives of ω and ϕ:

$$
\begin{aligned}
\nu_{rs}\lambda^r\lambda^s &= -\lambda_{rs}\nu^r\lambda^s &= \omega_{/1}\cos\phi, \\
\nu_{rs}\lambda^r\mu^s &= -\lambda_{rs}\nu^r\mu^s &= \omega_{/2}\cos\phi, \\
\nu_{rs}\lambda^r\nu^s &= -\lambda_{rs}\nu^r\nu^s &= \omega_{/3}\cos\phi;
\end{aligned}
\tag{4.1a}
$$

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or

$$\lambda_{rs}\mu^r\lambda^s = -\mu_{rs}\lambda^r\lambda^s = \omega_{/1}\sin\phi,$$
$$\lambda_{rs}\mu^r\mu^s = -\mu_{rs}\lambda^r\mu^s = \omega_{/2}\sin\phi, \qquad (4.1b)$$
$$\lambda_{rs}\mu^r\nu^s = -\mu_{rs}\lambda^r\nu^s = \omega_{/3}\sin\phi;$$

and

$$\mu_{rs}\nu^r\lambda^s = -\nu_{rs}\mu^r\lambda^s = -\phi_{/1}$$
$$\mu_{rs}\nu^r\mu^s = -\nu_{rs}\mu^r\mu^s = -\phi_{/2} \qquad (4.2)$$
$$\mu_{rs}\nu^r\nu^s = -\nu_{rs}\mu^r\nu^s = -\phi_{/3}$$

The double contractions appearing on the left-hand side of these equations may be immediately evaluated by using (2.2). This gives the following specializations:

$$\left(\omega_{/1},\omega_{/2},\omega_{/3}\right) = \left(-k_1\sec\phi, -t_1\sec\phi, \gamma_1\sec\phi\right), \qquad (4.3a)$$

or

$$\left(\omega_{/1},\omega_{/2},\omega_{/3}\right) = \left(\sigma_1\csc\phi, \sigma_2\csc\phi, \varepsilon_3\csc\phi\right), \qquad (4.3b)$$

and

$$\left(\phi_{/1},\phi_{/2},\phi_{/3}\right) = \left(-t_1, -k_2, \gamma_2\right), \qquad (4.4)$$

which we denote by $\{\omega\}$, $\{\omega^*\}$ and $\{\phi\}$ respectively.

It is interesting to note that Hotine failed to observe that the differential version of the geometrical equations (3.4) furnish *two values* for the ω-leg derivatives. He gave only $\{\omega\}$ and $\{\phi\}$, but not $\{\omega^*\}$. The two expressions $\{\omega\}$ and $\{\omega^*\}$ must be equal, and this implies a trivialization of *three* of the leg parameters, viz. *linear identities*

$$\sigma_1 = -k_1\tan\phi,$$
$$\sigma_2 = -t_1\tan\phi, \qquad (4.5a)$$
$$\varepsilon_3 = \gamma_1\tan\phi;$$

and *quadratic identities*:

$$\sigma_1 t_1 = k_1\sigma_2$$
$$\sigma_1\gamma_1 = -k_1\varepsilon_3 \qquad (4.5b)$$
$$\sigma_2\gamma_2 = -t_1\varepsilon_3$$

This phenomenon, which we call the ω-*degeneracy*, was first noted in [6] and yields the following expressions for the gradients of ω and ϕ:

$$\omega_r = \left(-k_1\lambda_r - t_1\mu_r + \gamma_1\nu_1\right)\sec\phi, \qquad (4.6a)$$

or

$$\omega_r = \left(\sigma_1\lambda_r + \sigma_2\mu_r + \varepsilon_3\nu_r\right)\csc\phi, \qquad (4.6b)$$

and

$$\phi_r = -t_1\lambda_r - k_2\mu_r + \gamma_2\nu_r. \qquad (4.7)$$

Equations (4.6a) and (4.7) appear in Hotine's analysis as [12.046] and [12.047].

An alternate version of these equations can be given in terms of the differentials of (ω,ϕ), i.e.

$$d\omega = \left(-k_1\theta_1 - t_1\theta_2 + \gamma_1\theta_3\right)\sec\phi, \qquad (4.8a)$$

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or

$$dw = (\sigma_1 \theta_1 + \sigma_2 \theta_2 + \varepsilon_3 \theta_3)\csc\phi, \qquad (4.8b)$$

and

$$d\phi = -t_1\theta_1 - k_2\theta_2 + \gamma_2\theta_3. \qquad (1.9)$$

In terms of absolute differentials, (3.4) becomes

$$
\begin{aligned}
D\lambda_r &= (\mu_r \sin\phi - \nu_r \cos\phi)\,dw \\
D_{\mu r} &= -\lambda_r \sin\phi\,dw - \nu_r d\phi \\
D\nu_r &= +\lambda_r \cos\phi\,dw + \mu_r d\phi,
\end{aligned} \qquad (4.10)
$$

and the ω-*degeneracy* is the observation that these equations do not provide a unique determination of dw. Of course,

$$dw = \mu^r D\nu_r = -\nu^r D\mu_r, \qquad (4.11)$$

so there is no ϕ-*degeneracy*. On the other hand, by contraction of (4.10) we have

$$
\begin{aligned}
dw &= \mu^r D\lambda_r \csc\phi &= -\lambda^r D\mu_r \csc\phi \\
dw &= \lambda^r D\nu_r \sec\phi &= -\nu^r D\lambda_r \sec\phi
\end{aligned} \qquad (4.12)
$$

A better derivation of $\{\omega\}$, $\{\omega^*\}$ and $\{\phi\}$ may be obtained by using (4.10), the Pfaffian expressions for dw and $d\phi$, and equating the corresponding coefficients of θ_1, θ_2 and θ_3. Then respectively $D\lambda$, $D\mu$ and $D\nu$ yield:

$$
\begin{aligned}
(\sigma_1\lambda_r + k_1\nu_r)\theta_1 &+ (\sigma_2\mu_r + t_1\nu_r)\theta_2 + (\varepsilon_3\mu_r - \gamma_1\nu_r)\theta_3 \\
&= \mu_r\left(\omega_{/1}\theta_1 + \omega_{/2}\theta_2 + \omega_{/3}\theta_3\right)\sin\phi \\
&- \nu_r\left(\omega_{/1}\theta_1 + \omega_{/2}\theta_2 + \omega_{/3}\theta_3\right)\cos\phi,
\end{aligned}
$$

$$
\begin{aligned}
(-\sigma_1\lambda_r + t_1\nu_r)\theta_1 &+ (-\sigma_2\lambda_r + k_2\nu_r)\theta_2 + (-\varepsilon_3\lambda_r - \gamma_2\nu_r)\theta_3 \\
&= -\lambda_r\left(\omega_{/1}\theta_1 + \omega_{/2}\theta_2 + \omega_{/3}\theta_3\right)\cos\phi \qquad (4.13) \\
&- \nu_r\left(\phi_{/1}\theta_1 + \phi_{/2}\theta_2 + \phi_{/3}\theta_3\right),
\end{aligned}
$$

$$
\begin{aligned}
(-k_1\lambda_r - t_1\mu_r)\theta_1 &+ (-t_1\lambda_r - k_2\mu_r)\theta_2 + (\gamma_1\lambda_r + \gamma_2\mu_r)\theta_3 \\
&= \lambda_r\left(\omega_{/1}\theta_1 + \omega_{/2}\theta_2 + \omega_{/3}\theta_3\right)\cos\phi \\
&+ \mu_r\left(\phi_{/1}\theta_1 + \phi_{/2}\theta_2 + \phi_{/3}\theta_3\right).
\end{aligned}
$$

Thus, we see that the $D\lambda$ equation gives $\{\omega\}$, $\{\omega^*\}$; $D\mu$ leads to $\{\omega^*\}$, $\{\phi\}$; while $D\nu$ yields $\{\omega\}$, $\{\phi\}$.

Each of these three versions of the *primary equations*, i.e. (4.3) and (4.4), or (4.6) and (4.7), or (4.8) and (4.9) or equivalently, (4.11) and (4.12), reveals that values $\phi = \pm\pi/2$ are inadmissible. These are the familiar polar singularities, but moreover the value $\phi = 0$ must be excluded! Indeed, for $\phi = \pm\pi/2$

by $\{\omega\}$, dw is undefined, while

by $\{\omega^*\}$, dw is well-defined;

and inversely for $\phi = 0$:

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by $\{\omega\}$, $d\omega$ is well-defined,

by $\{\omega^*\}$; $d\omega$ is undefined.

This requires the exclusion of these values and the ϕ range must be taken to be $-\pi/2 < \phi < 0$ and $0 < \phi < \pi/2$. The primary equations require no exclusion of ω values, however, one usually takes $0 \leq \omega \leq 2\pi$, or $0 < \omega \leq 2\pi$, to insure single-valuedness. The above excluded values furnish a second reason — apart from the occurrence of $\{\omega\}$ and $\{\omega^*\}$ *per se* — for the term ω-*degeneracy*.

The primary equations are so named since as we will see they play a paramount role in the theory of the construction of the (ω, ϕ, N) coordinate system. Our first indication of this is that the two primary equations we have obtained describe the variation of (ω, ϕ) on **S** or in $\mathbf{E_3}$. They justify our previous comment that, as Hotine said [page 71], $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are only roughly directed along the parallels and meridians of **S** respectively. Equations (4.6) and (4.7), or the corresponding differential expressions (4.8) and (4.9), give the precise alignments. The vectors $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ are aligned along these loci only when

$$t_1 = \gamma_1 = \gamma_2 = 0. \tag{4.14}$$

These conditions are satisfied identically for spherical polar coordinates $x^r = (\omega, \phi, r)$ in $\mathbf{E_3}$ and for $x^\alpha = (\omega, \phi)$ on the 2-sphere $\mathbf{S_2}$. A second even more fundamental role of the primary equations will be derived in Section 8.

5   Cartesian Integrability Conditions

As noted in Section 3, the first step in adopting the (ω, ϕ, N) coordinate system to the 3-leg $\left\{ \boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\nu} \right\}$ was the algebraic geometrical equations. As Hotine observed [page 71] since the constant 3-leg $\{\mathbf{A}, \mathbf{B}, \mathbf{C}\}$ is aligned along the Cartesian axes one has

$$x_r = A_r, \, y_r = B_r, \, z_r = C_r, \tag{5.1}$$

i.e. [12.009], where the subscripts on x, y, z denote partial derivatives with respect to the ambient coordinates x^r. Hence, upon contraction of these equations with dx^r, we obtain

$$
\begin{aligned}
dx &= -(\sin\omega)\,\theta_1 - (\cos\omega\sin\phi)\,\theta_2 + (\cos\omega\cos\phi)\,\theta_3, \\
dy &= (\cos\omega)\,\theta_1 - (\sin\omega\sin\phi)\,\theta_2 + (\sin\omega\cos\phi)\,\theta_3, \\
dz &= (\cos\phi)\,\theta_2 + (\sin\phi)\,\theta_3.
\end{aligned}
\tag{5.2}
$$

Since x, y, and z are functions, we must have

$$d^2x := d(dx) = 0, \, d^2y := d(dy) = 0, \, d^2z := d(dz) = 0, \tag{5.3}$$

and these equations constitute the *Cartesian integrability conditions* which the (ω, ϕ) coordinates and the Pfaffian forms θ_1, θ_2 and θ_3 must satisfy.

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Explicit calculation of the conditions (5.3) respectively yields

$$
\begin{aligned}
0 = & -(\cos\omega)\,d\omega \wedge \theta_1 - (\sin\omega)\,d\theta_1 \\
& + (\sin\omega\sin\phi)\,d\omega \wedge \theta_2 - (\cos\omega\sin\phi)\,d\phi \wedge \theta_2 - (\cos\omega\sin\phi)\,d\theta_2 \\
& - (\sin\omega\cos\phi)\,d\omega \wedge \theta_3 - (\cos\omega\sin\phi)\,d\phi \wedge \theta_3 + (\cos\omega\cos\phi)\,d\theta_3,
\end{aligned}
$$

$$
\begin{aligned}
0 = & -(\sin\omega)\,d\omega \wedge \theta_1 + (\cos\omega)\,d\theta_1 \\
& - (\cos\omega\sin\phi)\,d\omega \wedge \theta_2 - (\sin\omega\cos\phi)\,d\phi \wedge \theta_2 - (\sin\omega\sin\phi)\,d\theta_2 \\
& + (\cos\omega\cos\phi)\,d\phi \wedge \theta_3 - (\sin\omega\sin\phi)\,d\phi \wedge \theta_3 + (\sin\omega\cos\phi)\,d\theta_3,
\end{aligned}
\qquad (5.4)
$$

$$
\begin{aligned}
0 = & -(\sin\phi)\,d\phi \wedge \theta_2 + (\cos\phi)\,d\theta_2 \\
& + (\cos\phi)\,d\phi \wedge \theta_3 + (\sin\phi)\,d\theta_3.
\end{aligned}
$$

The leg equations corresponding to (5.4) may be obtained by using (2.9) when F is taken to be ω and ϕ respectively, and the structural equations (2.6). This yields *nine* equations which respectively occur as the coefficients of the linearly independent exterior products $\theta_1 \wedge \theta_2$, $\theta_3 \wedge \theta_1$ and $\theta_2 \wedge \theta_3$.

It is convenient to refer to these equations as *x-equations*, *y-equations*, and *z-equations* and in an obvious notation denote then as $(x_{1\wedge2})$, $(x_{3\wedge1})$, and $(x_{2\wedge3})$, etc. respectively. Upon cancellation of the common factors of $\sin\omega$ in the x-equations, $\cos\omega$ in the y-equations, and $\cos\phi$ in the z-equations we have

$$
\begin{aligned}
(x_{1\wedge2}) :\ & \omega_{/1}\sin\phi && + \omega_{/2}\cot\omega - \phi_{/1}\cot\omega\cos\phi \\
& && = \sigma_1 + \sigma_2\cot\omega\sin\phi, \\[2pt]
(x_{3\wedge1}) :\ & \omega_{/1}\cos\phi && - \omega_{/3}\cot\omega + \phi_{/1}\cot\omega\sin\phi \\
& && = -k_1 - (t_1 + \varepsilon_3)\cot\omega\sin\phi - \gamma_1\cot\omega\cos\phi, \\[2pt]
(x_{2\wedge3}) :\ & -\omega_{/2}\cos\phi && - \omega_{/3}\sin\phi - \phi_{/2}\cot\omega\sin\phi - \phi_{/3}\cot\omega\cos\phi \\
& && = t_1 - \varepsilon_3 + k_2\cot\omega\sin\phi + \gamma_2\cot\omega\sin\phi; \\[2pt]
(y_{1\wedge3}) :\ & -\omega_1\sin\phi && + \omega_{/2}\tan\omega - \phi_{/1}\tan\omega\cos\phi \\
& && = -\sigma_1 + \sigma_2\tan\omega\sin\phi, \\[2pt]
(y_{3\wedge1}) :\ & -\omega_1\cos\phi && - \omega_{/3}\tan\omega + \phi_{/1}\tan\omega\sin\phi \\
& && = -k_1 - (t_1 + \varepsilon_3)\tan\omega\sin\phi - \gamma_1\tan\omega\cos\phi, \\[2pt]
(y_{2\wedge3}) :\ & \omega_{/2}\cos\phi && + \omega_{/3}\sin\phi - \phi_{/2}\tan\omega\sin\phi + \phi_{/3}\tan\omega\cos\phi \\
& && = \varepsilon_3 - t_1 + k_2\tan\omega\sin\phi + \gamma_2\tan\omega\cos\phi;
\end{aligned}
\qquad (5.5)
$$

$$
(z_{1\wedge2}) :\ -\phi_{/1}\tan\phi = -\sigma_2,
$$

$$
(z_{3\wedge1}) :\ -\phi_1 = t_1 + \varepsilon_3 - \gamma_1\tan\phi,
$$

$$
(z_{2\wedge3}) :\ \phi_{/2} + \phi_{/3}\tan\phi = \gamma_2\tan\phi - k_2.
$$

These are automatically identically satisfied by virtue of the values of the leg paramenters given in $\{\omega\}$, $\{\omega^*\}$, $\{\phi\}$ and by using (4.5)! It would be tempting to believe that, as in our alternate derivation of the expressions for the covariant derivatives of the leg vectors given in Section 3 (see the equations displayed in (3.6)), that the system of x, y, and z-equations can be inverted to yield the expressions $\{\omega\}$, $\{\omega^*\}$ and $\{\phi\}$. This is not the case. The values of the leg derivatives of ω are thoroughly mixed in the above system. For example, $(x_{1\wedge2})$ is checked by using

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ω-derivatives from $\{\omega^*\}$, while $(x_{3\wedge 1})$ employs ω-derivatives from $\{\omega\}$. Moreover, as is easily seen from $(z_{1\wedge 2})$ and $(z_{3\wedge 1})$ we obtain the following ϕ-derivatives:

$$
\begin{aligned}
\phi_{/1} &= \sigma_2 - \cot\phi \\
\phi_{/1} &= -t_1 - \varepsilon_3 + \gamma_1 \tan\phi
\end{aligned}
$$

which leads to the identity

$$
\sigma_2 = -(t_1 + \varepsilon_3)\tan\phi + \gamma_1 \tan^2\phi
$$

which is quite intractable unless (4.5) is known.

6   Components of the Leg Vectors and the Line Element

The primary equations for ω and ϕ were given in Section 4, and now to obtain the (ω, ϕ, N) specializations of the contravariant and covariant components of the leg vectors we need a primary equation for N.

If N is the geopotential function, and the equipotential surface $\mathbf{S}$ is defined by putting $N = $ constant, then we have the *basic gradient equation*

$$
N_r = n\nu_r \tag{6.1}
$$

which expresses the well known property that the gradient of N with respect to x^r is collinear with the normal to $\mathbf{S}$. By hypothesis, ν is a unit vector, so the factor n is required to insure that $n^{-1}N_r$ will be of unit magnitude. Physically n is the magnitude of the *local gravity* along the plumblines Γ of the N-surface $\mathbf{S}$. Equation (6.1) is a general result, and upon employing the second part $x^3 := N$ of the Marussi Ansatz it yields

$$
\nu_r = n^{-1}\delta_r^3. \tag{6.2}
$$

Hence, we have the leg analogue of (6.1)

$$
\left(N_{/1}, N_{/2}, N_{/3}\right) = (0, 0, n) \tag{6.3}
$$

which is our final primary equation, while the Pfaffian version is given by

$$
dN = n\theta_3. \tag{6.4}
$$

Since each of the leg vectors is a tangent vector, they may respectively be expressed in the form dx^r/ds where s is an appropriate arc length along the respective three congruences of curves in $\mathbf{E}_3$. Upon making the specialization $x^r = (\omega, \phi, N)$ it follows that the leg derivatives in the corresponding directions are given by

$$
\begin{aligned}
\lambda^r &= \left(\omega_{/1}, \phi_{/1}, N_{/1}\right), \\
\mu^r &= \left(\omega_{/2}, \phi_{/2}, N_{/2}\right), \\
\nu^r &= \left(\omega_{/3}, \phi_{/3}, N_{/3}\right).
\end{aligned} \tag{6.5}
$$

By virtue of our (ω, ϕ, N) primary equations, these become

$$
\begin{aligned}
\lambda^r &= (-k_1 \sec\phi, -t_1, 0), \\
\mu^r &= (-t_1 \sec\phi, -k_2, 0), \\
\nu^r &= (\gamma_1 \sec\phi, \gamma_2, n),
\end{aligned}
\tag{6.6}
$$

i.e. [12.029], [12.030] and [12.034], which are the *canonical components of the contravariant leg vectors* in this coordinate system.

The corresponding canonical covariant components of the leg vectors were derived by Hotine by a rather lengthy process [pages 74-75] involving systematically writing out the components of [2.07], viz.

$$
\lambda^r \lambda_s + \mu^r \mu_s + \nu^r \nu_s = \delta_s^r.
\tag{6.7}
$$

A quicker method is to observe that (6.7) is equivalent to the matrix equations

$$
\left\|
\begin{array}{c}
\lambda^1, \lambda^2, \lambda^3 \\
\mu^1, \mu^2, \mu^3 \\
\nu^1, \nu^2, \nu^3
\end{array}
\right\|
\left\|
\begin{array}{c}
\lambda_1, \mu_1, \nu_1 \\
\lambda_2, \mu_2, \nu_2 \\
\lambda_3, \mu_3, \nu_3
\end{array}
\right\|
=
\left\|
\begin{array}{c}
1, 0, 0 \\
0, 1, 0 \\
0, 0, 1
\end{array}
\right\|.
\tag{6.8}
$$

The matrices appearing on the left-hand side of this equation are non-singular, and since by (6.6) the entries in the first matrix are known we immediately obtain the following *canonical covariant components of the leg vectors*

$$
\begin{aligned}
\lambda_r &= \left(-k_2 K^{-1}\cos\phi, t_1 K^{-1}, (k_2\gamma_1 - t_1\gamma_2)n^{-1}K^{-1}\right) \\
\mu_r &= \left(t_1 K^{-1}\cos\phi, -k_1 K^{-1}, (k_1\gamma_2 - t_1\gamma_1)n^{-1}K^{-1}\right) \\
\nu_r &= \left(0, 0, n^{-1}\right).
\end{aligned}
\tag{6.9}
$$

For purely reference purposes we note that if (6.5) is rewritten in matrix form using (6.3) as

$$
\left\|
\begin{array}{c}
\lambda^1, \lambda^2, \lambda^3 \\
\mu^1, \mu^2, \mu^3 \\
\nu^1, \nu^2, \nu^3
\end{array}
\right\|
=
\left\|
\begin{array}{c}
\omega_{/1}, \phi_{/1}, 0 \\
\omega_{/2}, \phi_{/2}, 0 \\
\omega_{/3}, \phi_{/3}, n
\end{array}
\right\|,
\tag{6.10}
$$

then denoting its determinant by DET we have

$$
\mathrm{DET} = \left(\omega_{/1}\phi_{/2} - \omega_{/2}\phi_1\right)n;
\tag{6.11}
$$

and consequently

$$
\left\|
\begin{array}{c}
\lambda_1, \mu_1, \nu_1 \\
\lambda_2, \mu_2, \nu_2 \\
\lambda_3, \mu_3, \nu_3
\end{array}
\right\|
=
\frac{1}{\mathrm{DET}}
\left\|
\begin{array}{ccc}
\phi_{/2}n, & -\phi_{/1}n, & 0 \\
-\omega_{/2}n, & \omega_{/1}n, & 0 \\
c_1, & c_2, & c_3
\end{array}
\right\|
\tag{6.12}
$$

where

$$
\begin{aligned}
c_1 &:= \omega_{/2}\phi_{/3} - \omega_{/3}\phi_{/2}, \\
c_2 &:= \omega_{/3}\phi_{/1} - \omega_{/1}\phi_3, \\
c_3 &:= \omega_{/1}\phi_{/2} - \omega_{/2}\phi_{/1}.
\end{aligned}
$$

By virtue of (2.1) the (ω, ϕ, N) Pfaffian forms are given by

$$
\begin{aligned}
\theta_1 &= \left\{-k_2 \cos \phi\, d\omega + t_1 d\phi + (k_2 \gamma_1 - t_1 \gamma_2)\, dN/n\right\}/K, \\
\theta_2 &= \left\{t_1 \cos \phi\, d\omega - k_1 d\phi + (k_1 \gamma_2 - t_1 \gamma_1)\, dN/n\right\}/K, \\
\theta_3 &= dN/n.
\end{aligned}
\tag{6.13}
$$

The leg representation of the metric tensor g_{rs} is given by [2.08], i.e.

$$
g_{rs} = \lambda_r \lambda_s + \mu_r \mu_s + \nu_r \nu_s,
\tag{6.14}
$$

and, hence, the line element ds^2 of $\mathbf{E}_3$ in the (ω, ϕ, N) system is

$$
ds^2 = (\theta_1)^2 + (\theta_2)^2 + (\theta_3)^2.
\tag{6.15}
$$

The explicit values of g_{rs} are displayed in Hotine's [12.069] and need not be repeated here. The matrix version of (6.13) is given by

$$
\left\|\begin{array}{c} \theta_1 \\ \theta_2 \\ \theta_3 \end{array}\right\| = \mathcal{M} \left\|\begin{array}{c} d\omega \\ d\phi \\ dN \end{array}\right\|,
\tag{6.14}
$$

where the entries of the 3×3 matrix $\mathcal{M}$ are obvious from (6.13). Hence,

$$
\left\|\begin{array}{c} d\omega \\ d\phi \\ dN \end{array}\right\| = \mathcal{M}^{-1} \left\|\begin{array}{c} \theta_1 \\ \theta_2 \\ \theta_3 \end{array}\right\|,
\tag{6.15}
$$

which is a *fourth* version of the full set of primary equations. Explicitly (6.14) is

$$
\left\|\begin{array}{c} d\omega \\ d\phi \\ dN \end{array}\right\| = \left\|\begin{array}{ccc} -k_1 \sec \phi, & -t_1 \sec \phi, & \gamma_1 \sec \phi \\ -t_1, & -k_2, & \gamma_2 \\ 0, & 0, & n \end{array}\right\| \cdot \left\|\begin{array}{c} \theta_1 \\ \theta_2 \\ \theta_3 \end{array}\right\|
\tag{6.16}
$$

and the matrix $\mathcal{M}^{-1}$ is called by Grafarend [9] the *matrix of integrating factors*. He notes that one *cannot* take

$$
\theta_1 = dX, \quad \theta_2 = dY, \quad \theta_3 = dZ,
\tag{6.17}
$$

i.e. the θ_1, θ_2 and θ_3 are *not* perfect differentials, and he regards this observation as being "... one of the most fundamental formula of geodetic science ...". In Section 8 we will show

$$
\begin{aligned}
d\,(d\omega) &= d^2\omega &= 0 \\
d\,(d\phi) &= d^2\phi &= 0 \\
d\,(dN) &= d^2N &= 0
\end{aligned}
\tag{6.18}
$$

when θ_1, θ_2, θ_3 are general Pfaffian forms, and that, in effect, (6.18) constitutes a set of integrability conditions which reduce to the (ω, ϕ) expressions for the Lamé equations and the N-integrability conditions. Grafarend's result is that one cannot

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trivialize θ_1, θ_2, θ_3 to the choice (6.17) since then the resulting integrability conditions would not be satisfied.

The determinant g of the matrix $\|g_{rs}\|$ is readily evaluated by using Hotine's expressions, i.e. [12.069], and, hence

$$\sqrt{g} = n^{-1} K^{-1} \cos\phi. \tag{6.19}$$

As expected $\sqrt{g}$, and the matrices $\mathcal{M}$, $\mathcal{M}^{-1}$, admit *polar singularities* at $\phi = \pm\pi/2$, but are well-behaved at $\phi = 0$.

7 Digression on Spherical Polar Coordinates

In this section we examine the case of spherical polar coordinates $x^r = (\omega, \phi, r)$ and consider to what extent the (ω, ϕ, N) system may be regarded as a *generalized spherical polar coordinate system*.

First, we note that the definitions of the surface coordinates $x^\alpha = (\omega, \phi)$ in both systems are identical. Hence, all of our adaptation of (ω, ϕ) to the leg calculus — up to the beginning of Section 6 — hold for both systems. The ω-degeneracy and the (ω, ϕ)-primary equations are common to both systems. This is not familiar merely because the spherical polar system is not usually constructed by the leg-theoretic procedure we have followed. The initial generalization consists in the fact noted at the end of Section 4 that whereas in the polar spherical system the vectors λ, μ are exactly aligned along the parallels (ω-lines) and meridians (ϕ-lines) of S, this is only roughly the case for the (ω, ϕ, N) systems. Furthermore, as a consequence of (4.14) the plumblines Γ of S have $\chi = 0$, i.e. are straight radial lines, for the polar system this need not be the case for the (ω, ϕ, N) system (recall the definition of χ given in the discussion following equation (2.3)). Hence, for the polar system the r-surfaces are 2-spheres S_2: $r = $ constant, while in the (ω, ϕ, N) system the N-surfaces S are more general curved surfaces. The precise character of these N-surfaces is left unspecified in the Marussi-Hotine approach to differential geodesy. Indeed, they assumed that the determination of N was a task for physical — not differential — geodesy, and their use of the (ω, ϕ, N) coordinate system was purely of a descriptive nature designed to provide a convenient description of a given, but unspecified, geopotential field N. Thus, the second generalization involved is that whereas the x^3-surfaces in the polar system are restricted to be 2-spheres of constant radii, for the (ω, ϕ, N) system these surfaces are arbitrary — subject only to the requirements imposed by the demands of physical geodesy. It is known that if the S are *sphere-like* in the sense of the Pizzetti theorem, [10], then these are compact surfaces in E_3 whenever the Pizzetti inequality $R \leq 5R_m$ where R is an appropriated defined radius and R_m is the mean radius of the Earth. A more general result is that such *spherelike* surfaces are diffeomorphic, i.e. differentiability equivalent in a topological sense, to a family of 2-spheres $\sum$ concentric with the Earth, [11]. Thus, this would include spheroids and those S which are convex or have $K > 0$.

In Section 5 we considered the Cartesian Integrability conditions and showed that they were identically satisfied by virtue of the ω, ϕ-primary equations. We now

consider the analogous situation for the spherical polar system (ω, ϕ, r). The spherical polar line element is given by

$$ds^2 = r^2 \cos^2 \phi \, d\omega^2 + r^2 d\phi^2 + dr^2, \tag{7.1}$$

see [page 5], and hence, denoting the purely coordinate-based Pfaffian forms by ϑ_1, ϑ_2, ϑ_3, we have

$$\begin{aligned} \vartheta_1 &= r \cos \phi \, d\omega, \\ \vartheta_2 &= r d\phi, \\ \vartheta_3 &= dr; \end{aligned} \tag{7.2}$$

so

$$\begin{aligned} d\omega &= r^{-1} \sec \phi \, \vartheta_1, \\ d\phi &= r^{-1} \vartheta_2, \\ dr &= \vartheta_3. \end{aligned} \tag{7.3}$$

Then upon exterior differentiation, since

$$\begin{aligned} d\vartheta_1 &= r^{-1} \left\{ \tan \phi \vartheta_1 \wedge \vartheta_2 + \vartheta_3 \wedge \vartheta_1 \right\}, \\ d\vartheta_2 &= -r^{-1} \vartheta_2 \wedge \vartheta_3, \\ d\vartheta_3 &= 0, \end{aligned} \tag{7.4}$$

it is readily verified that — as expected — we have

$$d^2\omega = 0, \ d^2\phi = 0, \ d^2r = 0. \tag{7.5}$$

On the other hand, relative the leg based Pfaffian forms θ_1, θ_2, θ_3 we may write

$$\begin{aligned} \vartheta_1 &= r \cos \phi \, d\omega = r \left(\omega_{/1}\theta_1 + \omega_{/2}\theta_2 + \omega_{/3}\theta_3 \right) \cos \phi, \\ \vartheta_2 &= r d\phi = r \left(\phi_{/1}\theta_1 + \phi_{/2}\theta_2 + \phi_{/3}\theta_3 \right), \\ \vartheta_3 &= dr = \left(r_{/1}\theta_1 + r_{/2}\theta_2 + r_{/3}\theta_3 \right). \end{aligned} \tag{7.6}$$

This yields as our *polar spherical primary equations*

$$\begin{cases} \left(\omega_{/1}, \omega_{/2}, \omega_{/3} \right) &= \left(r^{-1} \sec \phi, 0, 0 \right) \\ \left(\phi_{/1}, \phi_{/2}, \phi_{/3} \right) &= \left(0, r^{-1}, 0 \right) \\ \left(r_{/1}, r_{/2}, r_{/3} \right) &= \left(0, 0, 1 \right) \end{cases} \tag{7.7}$$

which confirms the comments on the alignment of λ, μ, and ν with the parallel, meridian, and radial directions given at the end of Section 4.

Consequently, by the arguments given in Section 6 the canonical contravariant components of the leg vectors in the polar system are given by

$$\begin{aligned} \lambda^r &= \left(\omega_{/1}, \phi_{/1}, r_{/1} \right) &= \left(r^{-1} \sec \phi, 0, 0 \right) \\ \mu^r &= \left(\omega_{/2}, \phi_{/2}, r_{/2} \right) &= \left(0, r^{-1}, 0 \right) \\ \nu^r &= \left(\omega_{/3}, \phi_{/3}, r_{/3} \right) &= \left(0, 0, 1 \right) \end{aligned} \tag{7.8}$$

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and the canonical covariant components by

$$\begin{aligned}
\lambda_r &= (r\cos\phi, 0, 0) \\
\mu_r &= (0, r, 0) \\
\nu_r &= (0, 0, 1).
\end{aligned} \tag{7.9}$$

The latter equations are, of course, also immediate by the expressions given in (7.6).

An obvious question is why the ω-degeneracy has *not* appeared in the primary equations (7.7). The answer is that, as suggested earlier in this section, the ω-*degeneracy* occurs by virtue of the construction based on the geometrical equations (3.2). In other words, the above argument proves that this *degeneracy* is a consequence of our method for constructing the (ω, ϕ, r) system rather than a structural feature of this system. On the other hand, in the case of the (ω, ϕ, N) system the x^3 coordinate is not a *pure* coordinate as it is in the spherical polar system. Hence, since the only known construction of the (ω, ϕ, N) system is based on the geometrical equations, the ω-*degeneracy* is unavoidable — at least in this sense — on the basis of present knowledge.

8   The (ω, ϕ, N) Integrability Conditions

In Section 5 we considered the Cartesian integrability conditions:

$$d^2x = 0, \; d^2y = 0, \; d^2z = 0$$

and showed that they were identically satisfied by using the primary equations $\{\omega\}$, $\{\omega^*\}$, and $\{\phi\}$. In the present section we will consider whether the analogous situation holds for the (ω, ϕ, N) integrability conditions. This investigation completes that begun in our previous paper [6].

Our first step is to note that the N-integrability conditions are exceptional and have no analogue in the polar spherical system considered in Section 7. In that case $\vartheta_3 = dr$ and the only non-zero r-leg derivative was $r_{/3} = 1$. Consequently, $d^2r = 0$ was trivial.

For the (ω, ϕ, N)-system, we have (6.4) where n is an arbitrary smooth function — not a constant! — so $d^2n = 0$ yields

$$0 = dn \wedge \theta_3 + n \wedge d\theta_3. \tag{8.1}$$

Upon using the Pfaffian representation of dn, i.e. (2.9), and (2.8) this equation yields

$$\left(-n_{/1} + n\gamma_1\right)\theta_3 \wedge \theta_1 + \left(n_{/2} - n\gamma_2\right)\theta_2 \wedge \theta_3,$$

or

$$\begin{aligned}
n_{/1} &= n\gamma_1, \\
n_{/2} &= n\gamma_2.
\end{aligned} \tag{8.2}$$

This equation may be regarded as the final step in the adaption of the leg calculus to the (ω, ϕ, N) system and, as such, it is a definition of the leg parameters γ_1, γ_2, i.e.

$$\gamma_1 := (\log n)_{/1}, \ \gamma_2 := (\log n)_{/2}. \tag{8.3}$$

This condition amounts to a scaling of Γ was derived by Hotine [page 73] in another manner, and when it i⁻ imposed our previously general 3-leg is said to be a *Hotine 3-leg*. Note that (6.4) holds regardless of whether the identification $x^3 := N$ has been made, so logically the notion of a Hotine 3-leg is independent of the Marussi Ansatz! Finally, one should investigate the n-integrability conditions, i.e. $d^2 n = 0$, however, since they are not relevant to the theory of the (ω, ϕ, N) system we refer the curious reader to [8] for a complete discussion of this issue.

Our principal task is now to consider the integrability conditions:

$$\begin{aligned} d(d\omega) &:= d^2\omega = 0, \\ d(d\phi) &:= d^2\phi = 0, \end{aligned} \tag{8.4}$$

when $d\omega$ is given by $\{\omega\}$ $\{\omega^*\}$, and $d\phi$ by $\{\phi\}$. Exterior differentiation of (4.8a) upon cancellation of a common factor of $\sec \phi$ gives

$$\begin{aligned} \tan \phi d\phi \ \wedge \ &(-k_1 \theta_1 - t_1 \theta_2 + \gamma_1 \theta_3) \\ - \ &dk_1 \wedge \theta_1 - dt_1 \wedge \theta_2 + d\gamma_1 \wedge \theta_3 \\ - \ &k_1 d\theta_1 - t_1 d\theta_2 + \gamma_1 d\theta_3 = 0, \end{aligned} \tag{8.5}$$

and similarily for (4.8b) with a factor of $\csc \phi$

$$\begin{aligned} -\cot \phi d\phi \ \wedge \ &(\sigma_1 \theta_1 + \sigma_2 \theta_2 + \varepsilon_3 \theta_3) \\ + \ &d\sigma_1 \wedge \theta_1 + d\sigma_2 \wedge \theta_2 + d\varepsilon_3 \wedge \theta_3 \\ + \ &\sigma_1 \wedge d\theta_1 + \sigma_2 \wedge d\theta_2 + \varepsilon_3 d\theta_3 = 0, \end{aligned} \tag{8.6}$$

and finally for (4.9) we obtain

$$-dt_1 \wedge \theta_1 - dk_2 \wedge \theta_2 + d\gamma_2 \wedge \theta_3 - t_1 d\theta_1 - k_2 d\theta_2 + \gamma_2 d\theta_3 = 0. \tag{8.7}$$

By using the Pfaffian representation for the differentials (2.9), the coefficients of the exterior products $\theta_1 \wedge \theta_2$, $\theta_3 \wedge \theta_1$ and $\theta_2 \wedge \theta_3$ lead to the following expressions which we denote by $(\omega_\mathcal{A})$, $(\omega_\mathcal{A}^*)$ and $(\phi_\mathcal{A})$ for $\mathcal{A} = $ I, II, III as in [6]:

$$\begin{aligned} (\omega_{\mathrm{I}}): \quad & k_{1/2} - t_{1/1} = (K - k_1^2 - t_1^2) \tan \phi, \\ (\omega_{\mathrm{II}}): \quad & k_{1/3} + \gamma_{1/1} = k_1^2 + t_1^2 + \gamma_1^2 + (2t_1 \gamma_1 - k_1 \gamma_2) \tan \phi, \\ (\omega_{\mathrm{III}}): \quad & t_{1/3} + \gamma_{1/2} = 2Ht_1 + \gamma_1 \gamma_2 - [t_1 \gamma_2 + (k_1 - k_2)\gamma_1] \tan \phi, \\ (\omega_{\mathrm{I}}^*): \quad & \sigma_{1/2} - \sigma_{2/1} = \sigma_1^2 + \sigma_2^2 - (k_2 \sigma_1 - t_1 \sigma_2) \cot \phi, \\ (\omega_{\mathrm{II}}^*): \quad & \varepsilon_{3/1} - \sigma_{1/3} = -\varepsilon_3 \sigma_2 + (\sigma_1^2 + \sigma_2^2 + \varepsilon_3^2 - \varepsilon_3 t_1 - \sigma_1 \gamma_2) \cot \phi, \\ (\omega_{\mathrm{III}}^*): \quad & \varepsilon_{3/2} - \sigma_{2/3} = \varepsilon_3 \sigma_1 - k_2 \sigma_2 + \varepsilon_3 \gamma_2 + (\sigma_1 \sigma_2 - k_2 \varepsilon_3 - \sigma_2 \gamma_2) \cot \phi, \\ (\phi_{\mathrm{I}}): \quad & k_{2/1} - t_{1/2} = 2Ht_1 \tan \phi, \\ (\phi_{\mathrm{II}}): \quad & t_{1/3} + \gamma_{2/1} = 2Ht_1 + \gamma_1 \gamma_2 + k_2 \gamma_1 \tan \phi, \\ (\phi_{\mathrm{III}}): \quad & k_{2/3} + \gamma_{2/2} = k_2^2 + t_1^2 + \gamma_2^2 - t_1 \gamma_1 \tan \phi. \end{aligned} \tag{8.8}$$

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As shown in [6] these correspond to Lamé equations (2.8) which are equivalent to the vanishing of the leg components the Riemann-Christoffel curvature tensor. More precisely, if we denote the vanishing of a component R_{abcd} by (R_{abcd}) we have

$$
\begin{aligned}
(R_{3112}) &\iff (\omega_{\mathrm{I}}), \\
(R_{3131}) &\iff (\omega_{\mathrm{II}}), \\
(R_{3123}) &\iff (\omega_{\mathrm{III}});
\end{aligned}
$$

$$
\begin{aligned}
(R_{1212}) &\iff (\omega_{\mathrm{I}}^{*}), \\
(R_{1231}) &\iff (\omega_{\mathrm{II}}^{*}), \\
(R_{1223}) &\iff (\omega_{\mathrm{III}}^{*});
\end{aligned}
\tag{8.9}
$$

$$
\begin{aligned}
(R_{2312}) &\iff (\phi_{\mathrm{I}}), \\
(R_{2331}) &\iff (\phi_{\mathrm{II}}), \\
(R_{2323}) &\iff (\phi_{\mathrm{III}}).
\end{aligned}
$$

By the algebraic Bianchi identities, we have

$$
\begin{aligned}
(R_{1231}) &\iff (R_{3112}) &\Rightarrow (\omega_{\mathrm{II}}^{*}) &\iff (\omega_{\mathrm{I}}), \\
(R_{1223}) &\iff (R_{2312}) &\Rightarrow (\omega_{\mathrm{III}}^{*}) &\iff (\phi_{\mathrm{I}}), \\
(R_{3123}) &\iff (R_{2331}) &\Rightarrow (\omega_{\mathrm{III}}) &\iff (\phi_{\mathrm{II}});
\end{aligned}
\tag{8.10}
$$

and by virtue of (4.5) it is clear that we have the reductions:

$$
(\omega_{\mathrm{I}}^{*}) \to (\omega_{\mathrm{I}}), \quad (\omega_{\mathrm{II}}^{*}) \to (\omega_{\mathrm{II}}), \quad (\omega_{\mathrm{III}}^{*}) \to (\omega_{\mathrm{III}}).
\tag{8.11}
$$

Hence, the six independent Lamé equations consist of

$$
\begin{aligned}
&(\omega_{\mathrm{I}}), \ (\omega_{\mathrm{II}}), \ (\omega_{\mathrm{III}}); \\
&(\phi_{\mathrm{I}}), \ (\phi_{\mathrm{II}}), \ (\phi_{\mathrm{III}});
\end{aligned}
$$

which corresponds respectively to the respective conditions $d^2\omega = 0$, $d^2\phi = 0$, viz. the ω and ϕ commutators.

9 Gaussian Differential Geometry of N-surfaces

In this section we use Gaussian Differential Geometry to argue that in a sense the (ω, ϕ, N) system can be regarded as a *generalized spherical polar coordinate system* in $\mathbf{E}_3$. Part of this is obvious since (ω, ϕ) play identical roles on both the 2-sphere $\mathbf{S}_2$ and the N-surface $\mathbf{S}$. The problem is to reconcile the difference between the radial coordinate r and the smooth, but unspecified, geopotential function N. As noted in Section 7, in the Marussi-Hotine approach to differential geodesy, the function N must be *a priori* specified since formally its determination lies *outside* the province of their theory.

Our starting point is the fact that an N-surface $\mathbf{S}$ is defined by the condition

$$
\theta_3 = 0
\tag{9.1}
$$

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and that the first and second basic forms of this surface are given by

$$I := (\theta_1)^2 + (\theta_2)^2 \tag{9.2}$$

$$II := \theta_1 \omega_{31} - \theta_2 \omega_{23} \tag{9.3}$$

where in the latter expression the connection 1-forms (2.7) are subject to (9.1). An equivalent version of (9.3) is

$$II = k_1 (\theta_1)^2 + t_1 (\theta_1\theta_2 + \theta_2\theta_1) + k_2 (\theta_2)^2. \tag{9.4}$$

By using the corresponding specializations of (6.10) we have

$$\begin{aligned} \theta_1 &= \{-k_2 \cos\phi d\omega + t_1 d\phi\} / K \\ \theta_2 &= \{t_1 \cos\phi d\omega - k_1 d\phi\} / K \end{aligned} \tag{9.5}$$

and I becomes

$$I = \left\{ \left(k_2^2 + t_1^2\right) \cos^2\phi d\omega^2 - 4Ht_1 \cos\phi d\omega d\phi + \left(k_1^2 + t_1^2\right) d\phi^2 \right\} / K^2, \tag{9.6}$$

while for $\mathbf{S}_2 : r = r_0$ where r_0 is a positive constant

$$I = r_0^2 \cos^2\phi d\omega^2 + r_0^2 d\phi^2. \tag{9.7}$$

The difference between (9.6) and (9.7) is striking, however, can be resolved by considering the five Hotine curvature parameters. It is convenient to display them in a quintuple

$$\langle k_1, k_2, t_1, \gamma_1, \gamma_2 \rangle.$$

For an $\mathbf{S}$, in general, all five entries are non-zero and non-trivial, while for $\mathbf{S}_2$ written in the usual (ω, ϕ) parametrization one has

$$\langle -1/r_0, -1/r_0, 0, 0, 0 \rangle.$$

However, for $\mathbf{S}$ one can obtain a nicer form of I by introducing the *radii of curvatures*

$$\begin{aligned} \rho_1 &:= (k_1 + t_1)^{-1} \\ \rho_2 &:= (k_2 + t_2)^{-1} = (k_2 - t_1)^{-1} \end{aligned} \tag{9.8}$$

since $t_2 = -t_1$. Then we have

$$K = (\rho_1\rho_2)^{-1} \tag{9.9}$$

and, hence

$$\begin{aligned} k_1^2 + t_1^2 + 2k_1t_1 &= \rho_1^{-2} \\ k_2^2 + t_1^2 - 2k_2t_1 &= \rho_2^{-2} \end{aligned} \tag{9.10}$$

and (9.6) becomes

$$I = \left\{ \left(\rho_2^{-2} + 2k_2t_1\right) \cos^2\phi d\omega^2 - 4Ht_1 \cos\phi d\omega d\phi + \left(\rho_1^{-2} - 2k_1t_1\right) d\phi^2 \right\} (\rho_1\rho_2)^2. \tag{9.11}$$

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On **S** one may choose without loss of generality

$$t_1 = 0, \tag{9.12}$$

which amounts to the selection of $\boldsymbol{\lambda}$, $\boldsymbol{\mu}$, as *principal directions* on **S**. This may be done by a rotation of the Hotine 3-leg $\{\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\nu}\}$ around $\boldsymbol{\nu}$. Then $\boldsymbol{\lambda}$, $\boldsymbol{\mu}$ become tangents to the ω-curves, ϕ-curves:

$$\boldsymbol{\lambda} = -k_1 \sec \phi \frac{\partial}{\partial \omega}, \quad \boldsymbol{\mu} = -k_2 \frac{\partial}{\partial \phi},$$

while

$$\boldsymbol{\nu} = \gamma_1 \sec \phi \frac{\partial}{\partial \omega} + \gamma_2 \frac{\partial}{\partial \phi} + n \frac{\partial}{\partial N}.$$

Thus, k_1, k_2 reduce to the *principal curvatures* κ_1, κ_2, and ρ_1, ρ_2 become the *radii* of the *principal curvatures*. Hence, the choice (9.12) permits the simplification of (9.11) to

$$\mathrm{I} = \rho_1^2 \cos^2 \phi d\omega^2 + \rho_2^2 d\phi^2. \tag{9.13}$$

In this form the analogy between (9.7) for $\mathbf{S}_2 : r = r_0$ and (9.3) for $\mathbf{S} : N = \text{constant}$ is clear. In passing from $\mathbf{S}_2$ to $\mathbf{S}$ one has replaced the *constant* r_0 by a pair of *variable* radii ρ_1 and ρ_2!

A similar argument can be given by considering II. For $\mathbf{S}_2$ we have

$$\mathrm{II} = -r_0 \cos^2 \phi d\omega^2 - r_0 d\phi^2, \tag{9.14}$$

while for (9.12) in the case of **S** one has

$$\mathrm{II} = \rho_1 \cos^2 \phi d\omega^2 + \rho_2 d\phi^2. \tag{9.15}$$

The sign differences between these expressions is inconsequential since by virtue of our outward orientation of **S**, κ_1, κ_2 and, thus, ρ_1, ρ_2, are negative. Hence, once again, our prescription given above for passing from $\mathbf{S}_2$ to **S** is valid.

These considerations lead us to suggest that $x^r := (\omega, \phi, N)$ may be regarded as a *generalized spherical polar coordinate system* in $\mathbf{E}_3$. Another aspect of this generalization is that although in $\mathbf{E}_3$

$$ds^2 = r^2 \cos^2 \phi d\omega^2 + r^2 d\phi^2 + dr^2 \tag{9.16}$$

is a triply orthogonal curvilinear coordinate system, i.e. $\mathbf{S}_2 : r = r_0$ is a member of a triply orthogonal system of surfaces, the corresponding situation is not true for $\mathbf{S} : N = \text{constant}$ in $\mathbf{E}_3$. This may be seen by examining Hotine's components for g_{rs} [12.069] in the (ω, ϕ, N) system subject to the restriction (9.12):

$$\begin{aligned}
ds^2 &= \left\{ k_2^2 \cos^2 \phi d\omega^2 + k_1^2 d\phi^2 \right. \\
&\quad + \left[\frac{(\gamma_1^2 k_2^2 + \gamma_2^2 k_1^2) + K^2}{n^2} \right] dN^2 \right\}/K \\
&\quad - 2 \left\{ [\gamma_1 k_2^2] d\omega dN \right\} / (n K^2 \sec \phi) \\
&\quad - 2 \left\{ [\gamma_2 k_1^2] d\phi dN \right\} / (n K^2).
\end{aligned} \tag{9.17}$$

Hence, (ω, ϕ, N) is not a triply orthogonal curvilinear coordinate system in $\mathbf{E}_3$, unless $\gamma_1 = \gamma_2 = 0$, which is equivalent to $\chi = 0$, viz. the normal congruence of curves consists of straight lines, which is an inadmissible specialization of the (ω, ϕ, N) system.

We now examine the conditions that an N-surface be a well-defined surface in $\mathbf{E}_3$. This requires examining the determinants $a := \det \|a_{\alpha\beta}\|$ and $b := \det \|b_{\alpha\beta}\|$ of the surface tensors appearing in I and II. Clearly,

$$
\begin{aligned}
a &= (\rho_1 \rho_2)^2 \cos \phi = K^{-2} \cos^2 \phi \\
b &= \rho_1 \rho_2 \cos^2 \phi = K^{-1} \cos^2 \phi
\end{aligned}
\tag{9.18}
$$

so $a > 0$, while $b \neq 0$ has variable sign, for $\phi \neq \pm \pi/2$. As expected, one has $K = b/a$. The Fundamental Theorem of Surfaces then states that modulo these conditions, a surface is uniquely defined in $\mathbf{E}_3$ up to position, i.e. a rigid motion in $\mathbf{E}_3$, whenever the equations of the Gauss and Codazzi are satisfied. The Gauss equations amount merely to (9.9), or $K = b/a$, and trivially hold. The Codazzi equations are less trivial and require that

$$
b_{\alpha\beta\gamma} - b_{\alpha\gamma\beta} = 0,
\tag{9.19}
$$

i.e. [6.21], where the final subscript indicates covariant differentiation.

The leg calculus version of this equation is given by

$$
\begin{aligned}
\sigma_1 (k_1 - k_2) &= k_{1/2} - t_{1/1} - 2t_1 \sigma_2, \\
\sigma_2 (k_1 - k_2) &= k_{2/1} - t_{1/2} + 2t_1 \sigma_1,
\end{aligned}
\tag{9.20}
$$

(see [8.23] in a slightly different notation). Using the $\{\omega^*\}$ and $\{\phi\}$ in terms of the (ω, ϕ) surface coordinates these become

$$
\begin{aligned}
k_{1/2} - t_{1/1} &= (K - k_1^2 - t_1^2) \tan \phi \\
k_{2/1} - t_{1/2} &= 2H t_1 \tan \phi
\end{aligned}
\tag{9.21}
$$

which are identical with (ω_{I}) and (ϕ_{I}) of (8.8).

10 Conclusions

In this concluding section we summarize and synthesize the results of the previous sections on the (ω, ϕ, N) system. The basic issue is how the leg calculus can be adapted to this coordinate system, and, in effect, whether these systems are compatible with each other. The leg calculus is a coordinate independent formalism and, as such, provides a lucid and searching picture of Gaussian Differential Geometry. Our starting point was a general 3-leg which was chosen to consist of a pair of tangent vectors to a surface $\mathbf{S}$ and a tangent to a congruence of curves Γ which is normal to $\mathbf{S}$. This leads to a set of three *basic leg equations* for the covariant derivatives of the leg vectors which are associated with *nine* leg parameters which completely describe the geometry of $\mathbf{S}$ and its orthogonal normal congruence. By virtue of its construction, one of these parameters t_2 may be eliminated, (i.e. $t_2 = -t_1$), without loss of generality and the curvatures K and H of $\mathbf{S}$ and χ, τ of Γ can be expressed in

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terms of the remaining eight independent leg parameters, six of which are *extrinsic* parameters, i.e. k_1, k_2, t_1, γ_1, γ_2, ε_3, with the remaining two, (i.e. σ_1, σ_2), being *intrinsic* parameters.

The construction of the (ω, ϕ, N)-system outlined by Hotine consists of three major sets of equations: two systems of *geometrical equations* (respectively of algebraic and differential form); a system of *primary equations* (given in three equivalent forms); and, finally, a system of *consistency equations*. Hotine's approach and construction of the (ω, ϕ, N)-system admirable and ingenious — as far as it goes! His presentation of the geometrical equations (in both the algebraic and differential forms) is complete. The difficulty begins with his primary equations where by an oversight he failed to notice that the first of his geometrical equations (in differential form) [12.014] admitted two non-trivial contracted products with the leg vectors. Hence, he obtained $\{\omega\}$ and $\{\phi\}$ but missed $\{\omega^*\}$, viz. the ω-degeneracy, as well as the fact that as a consequence *three* of the leg parameters are trivialized by the degeneracy. It was most uncharacteristic of him to have missed such a fact, and even more puzzling is the fact that nowhere in his work are the canonical differential equations exhibited. Our approach to the leg calculus is implicitly based on having these equations, and both of our derivations of the primary equations are impossible without having them in hand.

The geometrical equations (in algebraic form) provide the relationship between the Cartesian 3-leg $\{A, B, C\}$ of $\mathbf{E}_3$ and the general 3-leg $\left\{\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\nu}\right\}$, while the geometrical equations (in differential form) introduce the eight leg parameters into the analysis.

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